Math Logic: Model Theory & Computability
Lecture 23

Main lemma. Let τ be a signature. Every Henkin τ-theory H has a model.

Proof. Inspired by the last lemma (that H equates every term to a constant symbol) we may try to define the underlying set of our future model as just the set C of all constant symbols of τ. The only issue with this is that it might be that $(c_1 = c_2) \in H$ for two distinct constant symbols $c_1, c_2 \in C$. To fix this, we take as the underlying set $\tilde{C} := C/_\sim = \{[c]_\sim : c \in C\}$, where $\sim$ is the equivalence relation on C defined by

$$c_1 \sim c_2 \quad :\Leftrightarrow \quad (c_1 = c_2) \in H.$$

The axioms for equality (6) and maximal consistency of H ensures that $\sim$ is indeed an equivalence relation. For $\vec{a} := (a_1, a_2, \dots, a_n) \in C^n$, we denote

$$[\vec{a}]_\sim := ([a_1]_\sim, [a_2]_\sim, \dots, [a_n]_\sim).$$

We define a τ-structure $\underline{M} := (\tilde{C}, \tilde{\imath})$ as follows:

(i) $c^{\underline{M}} := [c]_\sim$ for each $c \in \mathrm{Const}(\tau) =: C$.

(ii) $f^{\underline{M}}([\vec{a}]_\sim) := [b]_\sim$ for each $f \in \mathrm{Funct}_k(\tau)$, $\vec{a} := (a_1, \dots, a_k)$, where $b \in C$ is such that $(f(a_1, a_2, \dots, a_k) = b) \in H$.

Proof of correctness. We first show that for each $\vec{a} \in C^k$ there is a $b \in C$ such that $(f(\vec{a}) = b) \in H$, but this is a special case of the previous lemma. Second, we need to show that the definition is $\sim$-invariant, i.e. doesn't depend on the representatives of $\sim$-classes. Suppose for $\vec{a}, \vec{c} \in C^k$ and $b, d \in C$ are such that $\vec{a} \sim \vec{c}$ (i.e. $a_i \sim c_i$), $(f(\vec{a}) = b) \in H$, and $(f(\vec{c}) = d) \in H$. By the axiom for equality and function symbol and maximal consistency of H, we have $(f(\vec{a}) = f(\vec{c})) \in H$ and hence also $(b = d) \in H$ by the transitivity-of-equality axiom and again maximality of H. $\square$

(iii) $R^{\underline{M}}([\vec{a}]_\sim)$ holds $:\Leftrightarrow$ $R(\vec{a}) \in H$, where $R \in \mathrm{Rel}_k(\tau)$ and $\vec{a} \in C^k$.

Proof of correctness. This follows from the axiom for equality and relation symbol, as well as the maximality of H. $\square$

Claim 1. For each τ-term t without variables and $b \in C$,
$$t^{\underline{M}} = [b]_\sim \text{ iff } (t = b) \in H.$$

Proof of Claim. We prove this by induction on the construction of t.

Case 1: $t := c$ for some $c \in C$. Then the claim follows from the def. of $\sim$.

Case 2: $t := f(t_1, \dots, t_k)$ for some $f \in \mathrm{Funct}_k(\tau)$ and τ-terms $t_1, \dots, t_k$ without variables. By induction we know that for all i, $t_i^{\underline{M}} = [b_i]_\sim$ iff $(t_i = b_i) \in H$, and there are b_i such that $(t_i = b_i) \in H$ by the previous lemma, so we also have $t_i^{\underline{M}} = [b_i]_\sim$. Then, by the def. of $\underline{M}$, $f^{\underline{M}}([b_1]_\sim, \dots, [b_k]_\sim) = [d]_\sim$ for some $d \in C$ such that $(f(b_1, \dots, b_k) = d) \in H$. Then by the transitivity-of-equality axiom and max. of H, $(d = c) \in H$, and hence also, $(f(t_1, \dots, t_k) = c) \in H$ by the axiom for equality and function symbol. $\square$

Claim 2. $\underline{M} \models \varphi$ iff $\varphi \in H$, for each τ-sentence φ.

Proof. We induct on the length/construction of φ.

Case 1: $\varphi := (t_1 = t_2)$ for some τ-terms t_1, t_2 w/o variables. Then let $[b_i]_\sim := t_i^{\underline{M}}$ so by Claim 1, $(t_i = b_i) \in H$. Thus, $(t_1 = t_2) \in H$ iff $(b_1 = b_2) \in H$ iff $[b_1]_\sim = [b_2]_\sim$ iff $t_1^{\underline{M}} = t_2^{\underline{M}}$ iff $\underline{M} \models t_1 = t_2$.

Case 2: $\varphi := R(t_1, \dots, t_k)$ for some τ-terms $t_1, \dots, t_k$ without variables. Then let $[b_i]_\sim := t_i^{\underline{M}}$, so by Claim 1, we have $(t_i = b_i) \in H$. Then $R(t_1, \dots, t_k) \in H$ iff $R(b_1, \dots, b_k) \in H$ iff (by the def. of $R^{\underline{M}}$) $R^{\underline{M}}([b_1]_\sim, \dots, [b_k]_\sim)$ holds iff $\underline{M} \models R(t_1, \dots, t_k)$.

Case 3: $\varphi := \neg \psi$ for some τ-sentence ψ. Then $\neg\psi \in H$ iff $\psi \notin H$ iff (by induction) $\underline{M} \nvDash \psi$ iff $\underline{M} \vDash \neg\psi$.

Case 4: $\varphi := \psi_1 \to \psi_2$ for some τ-sentences ψ_1, ψ_2. Then $(\psi_1 \to \psi_2) \in H$ iff $\neg\psi_1 \in H$ or $\psi_2 \in H$ (by maximality and consistency of H) iff (by induction) $\underline{M} \vDash \neg\psi_1$ or $\underline{M} \vDash \psi_2$ iff $\underline{M} \vDash (\psi_1 \to \psi_2)$.

Case 5: $\varphi := \exists v\, \psi$ for some extended τ-formula $\psi(v)$. Then $(\exists v \psi) \in H$ iff there is $c \in C$ such that $\psi(c/v) \in H$ ($\Rightarrow$ follows from the Henkinness of H and $\Leftarrow$ follows from HW9 Q2) iff there is $c \in C$ such that $\underline{M} \vDash \psi(c/v)$ iff there is $c \in C$ such that $\psi^{\underline{M}}(v)([c]_\sim)$ holds iff $\exists\, a \in \tilde{C}$ such that $\psi^{\underline{M}}(v)(a)$ holds iff $\underline{M} \vDash \exists v\, \psi$. $\square$

This finishes the proof of Main Lemma, and hence also of Gödel Completeness. $\square$

Complete and computable theories.

In the end of the 19^{th} and beginning of 20^{th} centuries, a demand arose (by Hilbert and others) to build a complete theory T for mathematics (set theory) or even just for arithmetic (i.e. $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$) such that the axioms of this theory are "easily recognizable." The latter term was formalized by saying that there is a computer program (equivalently, a Turing machine, a computable relation, etc) that recognizes the axioms of T, i.e. given a sentence φ, it returns YES if $\varphi \in T$ and NO, otherwise.

Nonexamples. (a) $Th(\underline{N})$, with $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, is complete by definition, but it is not even human recognizable, let alone by computers (recall Goldbach, Twin Primes).
(b) PA and ZFC are both computer recognizable but Gödel proved that

these are not complete theories. This theorem is known as Gödel's Incompleteness theorem, whose version (by Rosser) says that in fact any consistent theory T that is rich enough to "interpret" PA is either incomplete or non-computable (i.e. not computer recognizable.
In other words, "[illegible]".

We will sketch the proof of Gödel's Incompleteness, but let's discuss the same question about reducts of $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, namely: $\underline{N}_S := (\mathbb{N}, 0, S)$ and $\underline{N}_+ := (\mathbb{N}, 0, S, +)$.

Theorem. There is a complete computable theory T_S in the signature $(0, S)$ such that $\underline{N}_S \models T_S$, i.e. T_S is a computable axiomatization for $Th(\underline{N}_S)$.

Proof. Such a T_S was constructed in homework and proven to be complete using uncountable categoricity. ∎

Theorem (Presburger). There is a complete computable $(0, S, +)$-theory, namely $PresA := PA|_{(0,S,+)}$:= all axioms of PA that only use $0, S, +$, such that $\underline{N}_+ \models PresA$, i.e. $PresA$ is a computable axiomatization of $Th(\underline{N}_+)$.

Proof. PresA is call Presburger Arithmetic and the proof that it is complete as a $(0, S, +)$-theory uses the technique of quantifier elimination, which is beyond our course. In fact, the proof shows that not only PresA is complete but $Th(\underline{N}_+)$ is computable. ∎

The issue arises when we have both $+$ and $\cdot$ because this enables coding of tuples of natural numbers into single natural numbers (via the Chinese Remainder theorem), which in turn makes it possible to encode self-reference/diagonalization, hence Liar's Paradox: a sentence which says "I'm not provable from PA".